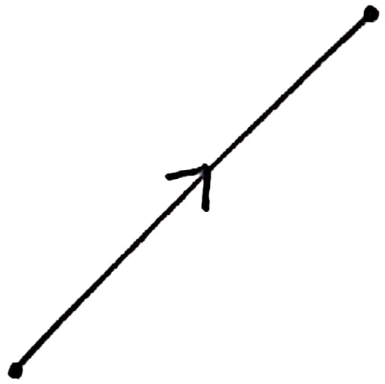

magnitude

direction

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

If $\vec{a} = t\hat{i} + (t^2+1)\hat{j}$

$$|\vec{a}| = \sqrt{t^2 + (t^2+1)^2}$$

NOTE - If $\vec{a}$ is a vector function in 't', then $|\vec{a}|$ is a scalar function in 't'

$u \rightarrow$ scalar function in 't'

$\vec{v} \rightarrow$ vector function in 't'

$$\frac{d(u\vec{v})}{dt} = u\frac{d\vec{v}}{dt} + \frac{du}{dt}\vec{v}$$

$\vec{u}, \vec{v} \rightarrow$ vector functions in 't'

$$\frac{d(\vec{u}\cdot\vec{v})}{dt} = \vec{u}\cdot\frac{d\vec{v}}{dt} + \vec{v}\cdot\frac{d\vec{u}}{dt}$$

Two Important Theorems In Vectors

#1. Condition For A Vector Function To Have Constant Magnitude

- ~~The~~
- The necessary and sufficient condition for a vector function $\vec{a}$ in variable 't' to have constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$

PROOF

TO PROVE THAT

$\vec{a}$ has constant magnitude $\iff \vec{a}\cdot\dfrac{d\vec{a}}{dt} = 0$

NECESSARY PART

Assume that $\vec{a}$ has constant magnitude

ie. $|\vec{a}| =$ a constant

$\therefore\ |\vec{a}|^2 = k$, a constant

$$\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0^\circ$$

ie. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

ie. $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$

ie. $\vec{a}\cdot\vec{a} = k$, a constant

$$\therefore \frac{d(\vec{a}\cdot\vec{a})}{dt} = \frac{d(k)}{dt}$$

ie. $\vec{a}\cdot\frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt}\cdot\vec{a} = 0$

$$\therefore 2\left(\vec{a}\cdot\frac{d\vec{a}}{dt}\right) = 0$$

$$\therefore \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$$

SUFFICIENT PART

Assume that $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$ .....(i)

We have $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

$$\therefore \frac{d(\vec{a} \cdot \vec{a})}{dt} = \frac{d(|\vec{a}|^2)}{dt}$$

ie. $\vec{a} \cdot \frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt} \cdot \vec{a} = 2|\vec{a}| \frac{d|\vec{a}|}{dt}$

ie. $0 + 0 = 2|\vec{a}| \frac{d|\vec{a}|}{dt}$ (By (i))

ie. $2|\vec{a}| \frac{d|\vec{a}|}{dt} = 0$

$$\therefore \frac{d|\vec{a}|}{dt} = 0$$

$\therefore |\vec{a}|$ is a constant

ie. $\vec{a}$ has constant magnitude

$$\frac{d x^2}{dx} = 2x$$

$$\frac{d(\sin x)^2}{dx} = 2\sin x . \cos x$$

$$\frac{d(\tan t)^2}{dx} = 2\tan t . \sec^2 t$$

#2. Condition For A Vector Function To Have Constant Direction

- The necessary and sufficient condition for a vector function $\vec{a}$ in variable 't' to have constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = 0$

Proof

To Prove That

$\vec{a}$ has constant direction $\iff \vec{a} \times \frac{d\vec{a}}{dt} = 0$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$\therefore \vec{a} = |\vec{a}|\,\hat{a} \quad \ldots\ldots (i)$$

$$\therefore \frac{d\vec{a}}{dt} = \frac{d\{|\vec{a}|\hat{a}\}}{dt}$$

$$\text{ie. } \frac{d\vec{a}}{dt} = |\vec{a}|\frac{d\hat{a}}{dt} + \frac{d|\vec{a}|}{dt}\hat{a}$$

$$\therefore \vec{a} \times \frac{d\vec{a}}{dt} = \vec{a} \times \left\{|\vec{a}|\frac{d\hat{a}}{dt} + \frac{d|\vec{a}|}{dt}\hat{a}\right\}$$

$$\text{ie. } \vec{a} \times \frac{d\vec{a}}{dt} = \vec{a} \times |\vec{a}|\frac{d\hat{a}}{dt} + \vec{a} \times \frac{d|\vec{a}|}{dt}\hat{a}$$

$$\text{ie. } \vec{a} \times \frac{d\vec{a}}{dt} = |\vec{a}|\hat{a} \times |\vec{a}|\frac{d\hat{a}}{dt} + |\vec{a}|\hat{a} \times \frac{d|\vec{a}|}{dt}\hat{a}$$

$$\vec{a} \times \frac{d\vec{a}}{dt} = |\vec{a}|^2 \left\{ \hat{a} \times \frac{d\hat{a}}{dt} \right\} + |\vec{a}| \frac{d|\vec{a}|}{dt} \left\{ \hat{a} \times \hat{a} \right\}$$

$$\text{ie. } \vec{a} \times \frac{d\vec{a}}{dt} = |\vec{a}|^2 \left\{ \hat{a} \times \frac{d\hat{a}}{dt} \right\} \quad \text{.....(i)}$$

Paste condit. -

NECESSARY CONDITION

Assume that $\vec{a}$ has constant direction

Since $\vec{a}$ and $\hat{a}$ share the same direction, $\hat{a}$ will also have a constant direction

But $|\hat{a}| = 1$, a constant

ie. $\hat{a}$ has constant magnitude as well as direction

ie. $\hat{a}$ is a constant vector

$$\therefore \frac{d\hat{a}}{dt} = 0$$

$$\therefore \text{By (i)} \quad \vec{a} \times \frac{d\vec{a}}{dt} = 0$$

SUFFICIENT CONDITION

Assume that $\vec{a} \times \frac{d\vec{a}}{dt} = 0$

$\therefore$ By (i) $|\vec{a}|^2 \left\{ \hat{a} \times \frac{d\hat{a}}{dt} \right\} = 0$

$$\therefore \hat{a} \times \frac{d\hat{a}}{dt} = 0 \quad \ldots\ldots \text{(ii)}$$

$|\hat{a}| = 1$

ie. $\hat{a}$ is a vector function with constant magnitude

$\therefore$ By Theorem #1.

$$\hat{a} \cdot \frac{d\hat{a}}{dt} = 0 \quad \ldots\ldots \text{(iii)}$$

(ii) & (iii) are possible only if $\frac{d\hat{a}}{dt} = 0$

ie. if $\hat{a}$ is a constant vector

$\therefore$ $\hat{a}$ has constant direction

Since $\hat{a}$ and $\vec{a}$ share the same direction, $\vec{a}$ has constant direction