FOURIER SERIES

$f(x) \rightarrow$ defined in an interval of length $2l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l} \int_{\bullet}^{\bullet} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{\bullet}^{\bullet} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_{\bullet}^{\bullet} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Where dots • • denote the lower & upper limit of the interval

pdf - Part 1

1. Obtain the Fourier series of the function $f(x) = x^2, \quad 0 < x < 2\pi$

$$2l = 2\pi - 0$$

ie. $2l = 2\pi$

$$\boxed{\therefore \; l = \pi}$$

Let the reqd. Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

ie. $$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \quad \text{—— (1)}$$

$$a_0 = \frac{1}{l}\int f(x)\,dx$$

ie. $$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

ie. $$a_0 = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi}$$

ie. $$a_0 = \frac{1}{3\pi}\left(x^3\right)_0^{2\pi}$$

$$\therefore \; a_0 = \frac{1}{3\pi}\left(8\pi^3 - 0\right)$$

$$\boxed{\text{ie. } a_0 = \frac{8}{3}\pi^2}$$

$$\int uv = u\bar{v} - u'\bar{\bar{v}} + u''\bar{\bar{\bar{v}}} - \cdots$$

$-$ stands for $\int$

$'$ stands for $\frac{d}{dx}$

PERFECT FOR $u \to$ POLYNOMIAL & $v \to$ sine/cosine/exp

$\cos n\pi = (-1)^n$ $\cos 2n\pi = 1$

$\sin n\pi = 0$

$$a_n = \frac{1}{l}\int f(x)\cos\frac{n\pi x}{l}\,dx$$

ie. $$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

ie. $$a_n = \frac{1}{\pi}\left[\left\{x^2\left(\frac{\sin nx}{n}\right)\right\}_0^{2\pi} - \left\{2x\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{2\pi} + \left\{2\left(\frac{-\sin nx}{n^3}\right)\right\}_0^{2\pi}\right]$$

$$\text{ie. } a_n = \frac{1}{\pi}\left[\frac{1}{n}\left\{x^2 \sin nx\right\}_0^{2\pi} + \frac{2}{n^2}\left\{x \cos nx\right\}_0^{2\pi} - \frac{2}{n^3}\left\{\sin nx\right\}_0^{2\pi}\right]$$

$$\text{ie. } a_n = \frac{1}{\pi}\left[\frac{1}{n}\left\{0-0\right\} + \frac{2}{n^2}\left\{2\pi \cos 2n\pi - 0\right\} - \frac{2}{n^3}\left\{0-0\right\}\right]$$

$$\text{ie. } a_n = \frac{1}{\cancel{\pi}} \times \frac{2}{n^2} \times 2\cancel{\pi}$$

$$\boxed{\text{ie. } a_n = \frac{4}{n^2}}$$

$$\text{Now, } b_n = \frac{1}{l}\int f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\text{ie. } b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$\text{ie. } b_n = \frac{1}{\pi}\left[\left\{x^2\left(\frac{-\cos nx}{n}\right)\right\}_0^{2\pi} - \left\{2x\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{2\pi} + \left\{2\left(\frac{\cos nx}{n^3}\right)\right\}_0^{2\pi}\right]$$

$$\text{ie. } b_n = \frac{1}{\pi}\left[\frac{-1}{n}\left\{x^2\cos nx\right\}_0^{2\pi} + \frac{2}{n^2}\left\{x\sin nx\right\}_0^{2\pi} + \frac{2}{n^3}\left\{\cos nx\right\}_0^{2\pi}\right]$$

$$\text{ie. } b_n = \frac{1}{\pi}\left[\frac{-1}{n}\left\{4\pi^2 - 0\right\} + 0 + \frac{2}{n^3}\left\{1-1\right\}\right]$$

$$\text{ie. } b_n = \frac{-1}{\pi}\,\frac{1}{n}\,4\pi^2$$

$$\text{ie. } b_n = -\frac{4\pi}{n}$$

$$\therefore \text{ By (1), } x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$\text{ie. } x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx$$

⑥

ie. $x^2 = \frac{4}{3}\pi^2 + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

e. $x^2 = \frac{4}{3}\pi^2 + 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right)$

$- 4\pi \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right)$