HALF RANGE SERIES

Consider a function $f(x)$ defined in $[0, l]$.

#1. Extend $f(x)$ in to an even function in $[-l, l]$

Then the fourier series of $f(x)$ will be a cosine series, given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l} \int_0^l f(x)\,dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$$

#2. Extend $f(x)$ into an odd function in $[-l, l]$.

Then the Fourier series of $f(x)$ will be a sine series, given by,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$$

1. $\left\{\right.$ Obtain the half range cosine series for $f(x) = x$, in the interval $0 \leq x \leq \pi$

$l = \pi - 0 = \pi$

Let the reqd. series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

ie. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \cdots (i)$

$$a_0 = \frac{2}{l}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left(x^2\right)_0^{\pi}$$

$$= \frac{1}{\pi}\left(\pi^2 - 0\right)$$

$\boxed{\text{ie. } a_0 = \pi}$

Now, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos \frac{n\pi x}{\pi}\,dx$

$$\text{ie. } a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$\text{ie. } a_n = \frac{2}{\pi}\left[\left\{x\left(\frac{\sin nx}{n}\right)\right\}_0^{\pi} - \left\{1.\left(\frac{-\cos nx}{n^2}\right)\right\}_0^{\pi}\right]$$

$$\text{ie. } a_n = \frac{2}{\pi}\left[\frac{1}{n}\left\{x \sin nx\right\}_0^{\pi} + \frac{1}{n^2}\left\{\cos nx\right\}_0^{\pi}\right]$$

$$\text{ie. } a_n = \frac{2}{\pi}\left[\frac{1}{n}\left\{0-0\right\} + \frac{1}{n^2}\left\{(-1)^n - 1\right\}\right]$$

$$\text{ie. } a_n = \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\}$$

$$\therefore\ x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\} \cos nx$$

2. Find the half range Fourier sine series for $f(x) = e^{2x}, \quad 0 < x < \pi$

Let the required series be

$$f(x) = \sum_{n=\phi}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$l = \pi$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

i.e. $$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots\ldots (i)$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} e^{2x} \sin nx\, dx$$

i.e. $b_n = \frac{2}{\pi}\left[\frac{e^{2x}}{2^2+n^2}\left\{2\sin nx - n\cos nx\right\}\right]_0^{\pi}$

i.e. $b_n = \frac{2}{\pi(4+n^2)}\left[e^{2x}\left\{2\sin nx - n\cos nx\right\}\right]_0^{\pi}$

i.e. $b_n = \frac{2}{\pi(4+n^2)}\left[e^{2\pi}\left\{0 - n(-1)^n\right\} - e^{0}\left\{0-n\right\}\right]$

i.e. $b_n = \frac{2}{\pi(4+n^2)}\left\{n - ne^{2\pi}(-1)^n\right\}$

i.e. $b_n = \frac{2n}{\pi(4+n^2)}\left\{1 - e^{2\pi}(-1)^n\right\}$

$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2n\left\{1-(-1)^n e^{2\pi}\right\}}{\pi(4+n^2)} \sin nx$